

On the Determinants and Inverses of Circulant Matrices with Pell and Pell-Lucas Numbers

Durmuş Bozkurt* & Fatih Yılmaz†

Department of Mathematics, Science Faculty,
Selçuk University, 42075 Kampus, Konya, Turkey

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Abstract

Let $\mathbb{P} = \text{circ}(P_1, P_2, \dots, P_n)$ and $\mathbb{Q} = \text{circ}(Q_1, Q_2, \dots, Q_n)$ be $n \times n$ circulant matrices where P_n and Q_n are n th Pell and Pell-Lucas numbers, respectively. The determinants of the matrices \mathbb{P} and \mathbb{Q} were expressed by the Pell and Pell-Lucas numbers. After, we prove that the matrices \mathbb{P} and \mathbb{Q} are the invertible for $n \geq 3$ and then the inverses of the matrices \mathbb{P} and \mathbb{Q} are derived.

1 Introduction

Circulant matrices have a widely range of application in signal processing, coding theory, image processing, digital image disposal, design of self-regress and so on. Also, numerical solutions of the certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems $Cx = b$ with C a circulant matrix [9-11].

The $n \times n$ circulant matrix $C_n = \text{circ}(c_0, c_1, \dots, c_{n-1})$, associated with the numbers c_0, c_1, \dots, c_{n-1} , is defined by

$$C_n := \begin{bmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \dots & c_0 & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{bmatrix}. \quad (1)$$

Circulant matrices have a wide range of applications, for examples in signal processing, coding theory, image processing, digital image disposal, self-regress design and so on. Numerical solutions of the certain types of elliptic

*e-mail: dbozkurt@selcuk.edu.tr

†e-mail: fyilmaz@selcuk.edu.tr

and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [9,10].

The eigenvalues and eigenvectors of C_n are well-known [4,14]:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{jk}, \quad j = 0, 1, \dots, n-1,$$

where $\omega := \exp(\frac{2\pi i}{n})$ and $i := \sqrt{-1}$ and the corresponding eigenvectors

$$x_j = (1, \omega^j, \omega^{2j}, \dots, \omega^{(n-1)j}), \quad j = 0, 1, \dots, n-1.$$

Thus we have the determinants and inverses of nonsingular circulant matrices [1,3,4,14]:

$$\det(C_n) = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} c_k \omega^{jk} \right)$$

and

$$C_n^{-1} = \text{circ}(a_0, a_1, \dots, a_{n-1})$$

where $a_k = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j \omega^{-jk}$, and $k = 0, 1, \dots, n-1$ [4]. When n is getting large, the above the determinant and inverse formulas are not very handy to use. If there is some structure among c_0, c_1, \dots, c_{n-1} , we may be able to get more explicit forms of the eigenvalues, determinants and inverses of C_n . Recently, studies on the circulant matrices involving interesting number sequences appeared. In [1] the determinants and inverses of the circulant matrices $A_n = \text{circ}(F_1, F_2, \dots, F_n)$ and $B_n = \text{circ}(L_1, L_2, \dots, L_n)$ are derived where F_n and L_n are n th Fibonacci and Lucas numbers, respectively. In [2] the r -circulant matrix is defined and its norms was computed. The norms of Toeplitz matrices [??] involving Fibonacci and Lucas numbers are obtained [5]. Miladinovic and Stanimirovic [6] gave an explicit formula of the Moore-Penrose inverse of singular generalized Fibonacci matrix. Lee and et al. found the factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices [7].

The Pell and Pell-Lucas sequences $\{P_n\}$ and $\{Q_n\}$ are defined by $P_n = 2P_{n-1} + P_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ with initial conditions $P_0 = 0$ and $P_1 = 1$ and $Q_0 = 2$ and $Q_1 = 2$ for $n \geq 2$, respectively. The Pell numbers are used to approximate the square root of two, to find square triangular numbers, to construct integer approximations to the right isosceles triangle, and to solve certain combinatorial enumeration problems. Let $\mathbb{P} = \text{circ}(P_1, P_2, \dots, P_n)$ and $\mathbb{Q} = \text{circ}(Q_1, Q_2, \dots, Q_n)$. The aim of this paper is to establish some useful formulas for the determinants and inverses of \mathbb{P} and \mathbb{Q} using the nice properties of the Pell and Pell-Lucas numbers. Matrix decompositions are derived for \mathbb{P} and \mathbb{Q} in order to obtain the results.

2 Determinants of circulant matrices with the Pell and Pell-Lucas numbers

Recall that $\mathbb{P} = \text{circ}(P_1, P_2, \dots, P_n)$ and $\mathbb{Q} = \text{circ}(Q_0, Q_1, \dots, Q_{n-1})$, i.e. where P_k and Q_k are k th Pell and Pell-Lucas numbers, respectively, with the recurrence relations $P_k = 2P_{k-1} + P_{k-2}$, $Q_k = 2Q_{k-1} + Q_{k-2}$, the initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = 2$ and $Q_1 = 2$ ($k \geq 2$). Let α and β be the roots of $x^2 - 2x - 1 = 0$. Using the Binet formulas [11, p. 142] for the sequences $\{P_n\}$ and $\{Q_n\}$, one has

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \quad (2)$$

and

$$Q_n = \alpha^n + \beta^n. \quad (3)$$

Theorem 1 *Let $n \geq 3$. Then*

$$\det(\mathbb{P}) = (P_1 - P_{n+1})^{n-2}(P_1 - 2P_n) + \sum_{k=2}^{n-1} [P_{k-1}P_n^{n-k}(P_1 - P_{n+1})^{k-2}] \quad (4)$$

where P_k is k th Pell number.

Proof. Obviously, $\det(\mathbb{P}) = 104$ for $n = 3$. It satisfies (4). For $n > 3$, we select the matrices M and N so that when we multiply \mathbb{P} with M on the left and N on the right we obtain a special Hessenberg matrix that have nonzero entries only on first two rows, main diagonal and subdiagonal:

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -2 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & 0 & 0 & \dots & -2 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ 0 & 1 & -2 & -1 & \dots & 0 & 0 \end{bmatrix} \quad (5)$$

and

$$N := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{P_n}{P_1 - P_{n+1}}\right)^{n-2} & 0 & \dots & 0 & 1 \\ 0 & \left(\frac{P_n}{P_1 - P_{n+1}}\right)^{n-3} & 0 & \dots & 1 & 0 \\ 0 & \left(\frac{P_n}{P_1 - P_{n+1}}\right)^{n-4} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \left(\frac{P_n}{P_1 - P_{n+1}}\right) & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Notice that we obtain the following equivalence:

$$\begin{aligned}
S &= M\mathbb{P}N \\
&= \begin{bmatrix}
1 & g'_n & P_{n-1} & P_{n-2} & P_{n-3} & \cdots & P_2 \\
& g_n & P_{n-2} & P_{n-3} & P_{n-4} & \cdots & P_1 \\
& & P_1 - P_{n+1} & & & & \\
& & -P_n & P_1 - P_{n+1} & & & 0 \\
& & & -P_n & P_1 - P_{n+1} & & \\
& 0 & & & -P_n & P_1 - P_{n+1} & \\
& & & & & -P_n & P_1 - P_{n+1}
\end{bmatrix}
\end{aligned}$$

and S is Hessenberg matrix, where

$$\begin{aligned}
g'_n &:= P_n + \sum_{k=2}^{n-1} P_k \left(\frac{P_n}{1 - P_{n+1}} \right)^{n-k}, \\
g_n &:= P_1 - 2P_n + \sum_{k=2}^{n-1} P_{k-1} \left(\frac{P_n}{P_1 - P_{n+1}} \right)^{n-k}.
\end{aligned}$$

Then we have

$$\det(S) = \det(M) \det(\mathbb{P}_n) \det(N) = (P_1 - P_{n+1})^{n-2} g_n.$$

Since

$$\det(M) = \det(N) = \begin{cases} 1, & n \equiv 1 \text{ or } 2 \pmod{4} \\ -1, & n \equiv 0 \text{ or } 3 \pmod{4} \end{cases}$$

for all $n > 3$,

$$\det(M) \det(N) = 1$$

and (4) follows. ■

Theorem 2 *Let $n \geq 3$. Then*

$$\det(\mathbb{Q}) = 2(2 - Q_{n+1})^{n-2} (2 - 3Q_n) + 2 \sum_{k=2}^{n-1} [(Q_{k+1} - 3Q_k)(Q_n - 2)^{n-k} (2 - Q_{n+1})^{k-2}]. \quad (6)$$

where Q_k is k th Pell-Lucas number.

Proof. Since $n \geq 3$, $\det(\mathbb{Q}) = 2464$ for $n = 3$, satisfies (6). For $n > 3$, we select the matrices K and L so that when we multiply \mathbb{Q} with K on the left and M

on the right we obtain a special Hessenberg matrix that have nonzero entries only on the first two rows, main diagonal and subdiagonal:

$$K := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -3 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & 0 & 0 & \dots & -2 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ 0 & 1 & -2 & -1 & \dots & 0 & 0 \end{bmatrix} \quad (7)$$

and

$$L := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{-2+Q_n}{Q_1-Q_{n+1}}\right)^{n-2} & 0 & \dots & 0 & 1 \\ 0 & \left(\frac{-2+Q_n}{Q_1-Q_{n+1}}\right)^{n-3} & 0 & \dots & 1 & 0 \\ 0 & \left(\frac{-2+Q_n}{Q_1-Q_{n+1}}\right)^{n-4} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \left(\frac{-2+Q_n}{Q_1-Q_{n+1}}\right) & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} U &= KQL \\ &= \begin{bmatrix} Q_1 & u'_n & Q_{n-1} & Q_{n-2} & \dots & Q_3 & Q_2 \\ & u_n & Q_n - 3Q_{n-1} & Q_{n-1} - 3Q_{n-2} & \dots & Q_4 - 3Q_3 & Q_3 - 3Q_2 \\ & & Q_1 - Q_{n+1} & & & & \\ & & 2 - Q_n & Q_1 - Q_{n+1} & & & \\ & & & 2 - Q_n & & & \\ & & & & \ddots & & \\ & & & & \ddots & Q_1 - Q_{n+1} & \\ & & & & & 2 - Q_n & Q_1 - Q_{n+1} \end{bmatrix} \end{aligned}$$

and U is Hessenberg matrix, where

$$u_n := Q_1 - 3Q_n + \sum_{k=2}^{n-1} (Q_{k+1} - 3Q_k) \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k},$$

$$u'_n := \sum_{k=2}^n Q_k \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k}.$$

Then we obtain

$$\det(U) = \det(K) \det(Q) \det(L) = Q_1(Q_1 - Q_{n+1})^{n-2} u_n.$$

Since

$$\det(K) = \det(L) = \begin{cases} 1, & n \equiv 1 \text{ or } 2 \pmod{4} \\ -1, & n \equiv 0 \text{ or } 3 \pmod{4} \end{cases}$$

for all $n > 3$,

$$\det(K) \det(L) = 1$$

and we have (6). ■

3 Inverses of \mathbb{P} and \mathbb{Q}

We will use the well-known fact that the inverse of a nonsingular circulant matrix is also circulant [14, p.84] [12, p.33] [4, p.90-91].

Theorem 3 *Let $\mathbb{P} = \text{circ}(P_1, P_2, \dots, P_n)$ is invertible when $n \geq 3$.*

Proof. From Theorem 1, we have $\det(\mathbb{P}) = 104 \neq 0$ and $\det(\mathbb{P}) = -18560 \neq 0$ for $n = 3$ and $n = 4$, respectively. Then \mathbb{P} is the invertible for $n = 3, 4$. Let $n \geq 5$. The Binet formula for Pell numbers gives $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$ where $\alpha + \beta = 2$, $\alpha\beta = -1$ and $\alpha - \beta = 2\sqrt{2}$. Then we have

$$\begin{aligned} u(\omega^k) &= \sum_{r=1}^n P_r \omega^{kr-k} = \sum_{r=1}^n \left(\frac{\alpha^r - \beta^r}{2\sqrt{2}} \right) \omega^{kr-k} = \frac{1}{2\sqrt{2}} \sum_{r=1}^n (\alpha^r - \beta^r) \omega^{kr-k} \\ &= \frac{1}{2\sqrt{2}} \left[\frac{\alpha(1 - \alpha^n)}{1 - \alpha\omega^k} - \frac{\beta(1 - \beta^n)}{1 - \beta\omega^k} \right], \quad (1 - \alpha\omega^k, 1 - \beta\omega^k \neq 0) \\ &= \frac{1}{2\sqrt{2}} \left(\frac{(\alpha - \beta) - (\alpha^{n+1} - \beta^{n+1}) + \alpha\beta\omega^k(\alpha^n - \beta^n)}{1 - \alpha\omega^k - \beta\omega^k + \alpha\beta\omega^{2k}} \right) \\ &= \frac{1 - P_{n+1} - P_n\omega^k}{1 - 2\omega^k - \omega^{2k}}, \quad k = 1, 2, \dots, n-1. \end{aligned}$$

If there existed ω^k ($k = 1, 2, \dots, n-1$) such that $u(\omega^k) = 0$, then we would have $1 - P_{n+1} - P_n\omega^k = 0$ for $1 - 2\omega^k - \omega^{2k} \neq 0$. Hence $\omega^k = \frac{1 - P_{n+1}}{P_n}$. It is well known that

$$\omega^k = \exp\left(\frac{2k\pi i}{n}\right) = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right) \quad (8)$$

where $i := \sqrt{-1}$. Since $\omega^k = \frac{1 - P_{n+1}}{P_n}$ is a real number, $\sin\left(\frac{2k\pi}{n}\right) = 0$ so that $\omega^k = -1$ for $0 < \frac{2k\pi}{n} < 2\pi$. However $x = -1$ is not a root of the equation $1 - P_{n+1} - P_n x = 0$ ($n \geq 5$), a contradiction. i.e. $u(\omega^k) \neq 0$ for any ω^k where $k = 1, 2, \dots, n-1$ and $n \geq 5$. Thus, the proof is completed by [1, Lemma 1.1]. ■

Lemma 4 Let $C = (c_{ij})$ be an $(n-2) \times (n-2)$ matrix defined by

$$c_{ij} = \begin{cases} P_1 - P_{n+1}, & i = j \\ -P_n, & i = j + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $C^{-1} = (c'_{ij})$ is given by

$$c'_{ij} = \begin{cases} \frac{P_n^{i-j}}{(P_1 - P_{n+1})^{i-j+1}}, & i \geq j \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $A := (a_{ij}) = CC^{-1}$. Clearly, $a_{ij} = \sum_{k=1}^{n-2} c_{ik} c'_{kj}$. When $i = j$, we have

$$a_{ii} = (P_1 - P_{n+1}) \cdot \frac{1}{P_1 - P_{n+1}} = 1.$$

If $i > j$, then

$$\begin{aligned} a_{ij} &= \sum_{k=1}^{n-2} c_{ik} c'_{kj} = c_{i,i-1} c'_{i-1,j} + c_{ii} c'_{ij} \\ &= -P_n \frac{P_n^{i-j-1}}{(P_1 - P_{n+1})^{i-j}} + (P_1 - P_{n+1}) \frac{P_n^{i-j}}{(P_1 - P_{n+1})^{i-j+1}} = 0; \end{aligned}$$

similar for $i < j$. Thus, $CC^{-1} = I_{n-2}$. ■

Theorem 5 Let the matrix \mathbb{P} be $\mathbb{P} = \text{circ}(P_1, P_2, \dots, P_n)$ ($n \geq 3$). Then the inverse of the matrix \mathbb{P} is

$$\mathbb{P}^{-1} = \text{circ}(p_1, p_2, \dots, p_n)$$

where

$$\begin{aligned} p_1 &= \frac{1}{g_n} \left(1 + \frac{2P_n^{n-3}}{(P_1 - P_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{P_{n-k} P_n^{k-1}}{(P_1 - P_{n+1})^k} \right) \\ p_2 &= \frac{1}{g_n} \left(-2 + \sum_{k=1}^{n-2} \frac{P_{n-k-1} P_n^{k-1}}{(P_1 - P_{n+1})^k} \right) \\ p_i &= -\frac{P_n^{i-3}}{g_n (P_1 - P_{n+1})^{i-2}}, \quad i = 3, 4, \dots, n \end{aligned}$$

$$\text{for } g_n = P_1 - 2P_n + \sum_{k=2}^{n-1} P_{k-1} \left(\frac{P_n}{P_1 - P_{n+1}} \right)^{n-k}.$$

Proof. Let

$$U = \begin{bmatrix} 1 & -g'_n & \frac{g'_n}{g_n}P_{n-2} - P_{n-1} & \frac{g'_n}{g_n}P_{n-3} - P_{n-2} & \dots & \frac{g'_n}{g_n}P_1 - P_2 \\ 0 & 1 & -\frac{P_{n-2}}{g_n} & -\frac{P_{n-3}}{g_n} & \dots & -\frac{P_1}{g_n} \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and $H = \text{diag}(1, g_n)$ where $g'_n = P_n + \sum_{k=2}^{n-1} P_k \left(\frac{P_n}{1-P_{n+1}} \right)^{n-k}$ and $g_n = P_1 - 2P_n + \sum_{k=2}^{n-1} P_k \left(\frac{P_n}{1-P_{n+1}} \right)^{n-k}$. Then we can write

$$M\mathbb{P}NU = H \oplus C$$

where $H \oplus C$ is the direct sum of the matrices H and C . Let $T = NU$. Then we have

$$\mathbb{P}^{-1} = T(H^{-1} \oplus C^{-1})M.$$

Since the matrix \mathbb{P} is circulant, its inverse is circulant from Lemma 1.1 [1, p.9791]. Let

$$\mathbb{P}^{-1} = \text{circ}(p_1, p_2, \dots, p_n).$$

Since the last row of the matrix T is

$$\left(0, 1, -\frac{P_{n-2}}{g_n}, -\frac{P_{n-3}}{g_n}, -\frac{P_{n-4}}{g_n}, \dots, -\frac{P_2}{g_n}, -\frac{P_1}{g_n} \right),$$

the last row components of the matrix \mathbb{P}^{-1} are

$$\begin{aligned} p_2 &= \frac{1}{g_n} \left(-2 + \sum_{k=1}^{n-2} \frac{P_{n-k-1}P_n^{k-1}}{(P_1 - P_{n+1})^k} \right) \\ p_3 &= -\frac{1}{g_n(P_1 - P_{n+1})} \\ p_4 &= -\frac{P_n}{g_n(P_1 - P_{n+1})^2} \\ p_5 &= -\frac{1}{g_n} \left(\sum_{k=1}^3 \frac{P_{4-k}P_n^{k-1}}{(P_1 - P_{n+1})^k} - 2 \sum_{k=1}^2 \frac{P_{3-k}P_n^{k-1}}{(P_1 - P_{n+1})^k} - \frac{P_1}{P_1 - P_{n+1}} \right) \\ &\vdots \\ p_n &= -\frac{1}{g_n} \left(\sum_{k=1}^{n-2} \frac{P_{n-k-1}P_n^{k-1}}{(P_1 - P_{n+1})^k} - 2 \sum_{k=1}^{n-3} \frac{P_{n-k-2}P_n^{k-1}}{(P_1 - P_{n+1})^k} - \sum_{k=1}^{n-4} \frac{P_{n-k-3}P_n^{k-1}}{(P_1 - P_{n+1})^k} \right) \\ p_1 &= \frac{1}{g_n} \left(1 + \frac{2P_n^{n-3}}{(P_1 - P_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{P_{n-k}P_n^{k-1}}{(P_1 - P_{n+1})^k} \right). \end{aligned}$$

where $g_n = P_1 - 2P_n + \sum_{k=2}^{n-1} P_{k-1} \left(\frac{P_n}{P_1 - P_{n+1}} \right)^{n-k}$. If $S_n^{(r)} = \sum_{k=1}^r \frac{P_{r-k+1} P_n^{k-1}}{(P_1 - P_{n+1})^k}$ ($r = 1, 2, \dots, n-2$), then we obtain

$$S_n^{(2)} - 2S_n^{(1)} = \frac{P_n}{(P_1 - P_{n+1})^2}$$

and

$$S_n^{(r+2)} - 2S_n^{(r+1)} - S_n^{(r)} = \frac{P_n^{r+1}}{(P_1 - P_{n+1})^{r+2}}, \quad r = 1, 2, \dots, n-4.$$

Hence, we have

$$\begin{aligned} \mathbb{P}^{-1} &= \frac{1}{g_n} \text{circ} \left(1 + 2S_n^{(n-2)} + S_n^{(n-3)}, -2 + S_n^{(n-2)}, -S_n^{(1)}, -S_n^{(2)} + 2S_n^{(1)}, \right. \\ &\quad \left. S_n^{(3)} - 2S_n^{(2)} - S_n^{(1)}, \dots, S_n^{(n-2)} - 2S_n^{(n-3)} - S_n^{(n-4)} \right) \\ &= \frac{1}{g_n} \text{circ} \left(1 + \frac{2P_n^{n-3}}{(P_1 - P_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{P_{n-k} P_n^{k-1}}{(P_1 - P_{n+1})^k}, -2 + \sum_{k=1}^{n-2} \frac{P_{n-k-1} P_n^{k-1}}{(P_1 - P_{n+1})^k}, \right. \\ &\quad \left. -\frac{1}{(P_1 - P_{n+1})}, -\frac{P_n}{(P_1 - P_{n+1})^2}, -\frac{P_n^2}{(P_1 - P_{n+1})^3}, \dots, -\frac{P_n^{n-3}}{(P_1 - P_{n+1})^{n-2}} \right). \end{aligned}$$

by the nice properties of Pell numbers. ■

A Hankel matrix $A = (a_{ij})$ is an $n \times n$ matrix such that $a_{i,j} = a_{i-1,j+1}$. It is closely related to the Toeplitz matrix in the sense that a Hankel matrix is an upside-down Toeplitz matrix.

Corollary 6 *Let the matrix M be as in (5). Then its inverse is*

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ B & C \end{bmatrix}$$

where the matrices

$$B = \begin{pmatrix} P_n & P_{n-1} & P_{n-2} & \dots & P_2 \end{pmatrix}_{(n-1) \times 1}^T$$

and C is an $(n-1) \times (n-1)$ Hankel matrix that it has the first row $[P_{n-1}, P_{n-2}, \dots, P_1]$ and the last column $[P_1, 0, \dots, 0]^T$.

Proof. The matrix M^{-1} is obtained easily by applying elementary row operations to the augmented matrix $[M : I_n]$. ■

Theorem 7 *The matrix $\mathbb{Q} = \text{circ}(Q_1, Q_2, \dots, Q_n)$ is invertible when $n \geq 3$.*

Proof. We show that $\det(\mathbb{Q}) = 2464 \neq 0$ and $\det(\mathbb{Q}) = -1247232 \neq 0$ by Theorem 2 for $n = 3, 4$. Then the matrix \mathbb{Q} is the invertible matrix for $n = 3, 4$. Let $n \geq 5$. The Binet formula for Jacobsthal-Lucas numbers yields $Q_n = \alpha^n + \beta^n$, where $\alpha + \beta = 2$ and $\alpha\beta = -1$. Then we have

$$\begin{aligned}
v(\omega^k) &= \sum_{r=1}^n Q_r \omega^{kr-k} = \sum_{r=1}^n (\alpha^r + \beta^r) \omega^{kr-k} \\
&= \frac{\alpha(1 - \alpha^n)}{1 - \alpha\omega^k} + \frac{\beta(1 - \beta^n)}{1 - \beta\omega^k}, \quad (1 - \alpha\omega^k, 1 - \beta\omega^k \neq 0) \\
&= \left(\frac{(\alpha + \beta) - (\alpha^{n+1} + \beta^{n+1}) + \alpha\beta\omega^k(\alpha^n + \beta^n) - 2\alpha\beta\omega^k}{1 - (\alpha + \beta)\omega^k + \alpha\beta\omega^{2k}} \right) \\
&= \frac{2 - Q_{n+1} + (2 - Q_n)\omega^k}{1 - 2\omega^k - \omega^{2k}}, \quad k = 1, 2, \dots, n-1.
\end{aligned}$$

We are going to show that there is no ω^k , $k = 1, 2, \dots, n-1$ such that $v(\omega^k) = 0$. If $2 - Q_{n+1} + (2 - Q_n)\omega^k = 0$ for $1 - 2\omega^k - \omega^{2k} \neq 0$, then $\omega^k = \frac{Q_{n+1}-2}{2-Q_n}$ would be a real number. By (8) we would have $\sin\left(\frac{2k\pi}{n}\right) = 0$ so that $\omega^k = -1$ for $0 < \frac{2k\pi}{n} < 2\pi$. However $x = -1$ is not a root of the equation $1 - Q_{n+1} + (2 - Q_n)x = 0$ ($n \geq 5$), a contradiction. i.e. $v(\omega^k) \neq 0$ for any ω^k , where $k = 1, 2, \dots, n-1$ and $n \geq 5$. Thus, the proof is completed by [1, Lemma 1.1]. ■

Lemma 8 *If the matrix $A = (a_{ij})_{i,j=1}^{n-2}$ is of the form*

$$a_{ij} = \begin{cases} Q_1 - Q_{n+1}, & i = j \\ 2 - Q_n, & i = j + 1 \\ 0, & \text{otherwise,} \end{cases}$$

then $A^{-1} = (a'_{ij})_{i,j=1}^{n-2}$ is given by

$$a'_{ij} = \begin{cases} \frac{(Q_n-2)^{i-j}}{(Q_1-Q_{n+1})^{i-j+1}}, & i \geq j \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $B := (b_{ij}) = AA^{-1}$ so that $b_{ij} = \sum_{k=1}^{n-2} a_{ik}a'_{kj}$. Clearly

$$b_{ii} = (Q_1 - Q_{n+1}) \cdot \frac{1}{Q_1 - Q_{n+1}} = 1.$$

If $i > j$, then

$$\begin{aligned}
b_{ij} &= \sum_{k=1}^{n-2} a_{ik}a'_{kj} = a_{i,i-1}a'_{i-1,j} + a_{ii}a'_{ij} \\
&= (2 - Q_n) \frac{(Q_n - 2)^{i-j-1}}{(Q_1 - Q_{n+1})^{i-j}} + (Q_1 - Q_{n+1}) \frac{(Q_n - 2)^{i-j}}{(Q_1 - Q_{n+1})^{i-j+1}} = 0;
\end{aligned}$$

similar for $i < j$. Thus $A^{-1}A = I_{n-2}$. ■

Theorem 9 Let $n \geq 3$. The inverse of the matrix \mathbb{Q} is

$$\mathbb{Q}^{-1} = \text{circ}(q_1, q_2, \dots, q_n)$$

where

$$\begin{aligned} q_1 &= \frac{1}{u_n} \left(1 - \frac{8(Q_n - 2)^{n-3}}{(Q_1 - Q_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{(Q_{n-k+2} - 3Q_{n-k+1})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \\ q_2 &= \frac{1}{u_n} \left(-3 + \sum_{k=1}^{n-2} (Q_{n-k+1} - 3Q_{n-k}) \left(\frac{(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \right) \\ q_m &= \frac{4(Q_n - 2)^{m-3}}{u_n(Q_1 - Q_{n+1})^{m-2}}, \quad m = 3, 4, \dots, n \end{aligned}$$

$$\text{for } u_n = Q_1 - 3Q_n + \sum_{k=2}^{n-1} (Q_{k+1} - 3Q_k) \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k}.$$

Proof. Let

$$\mathbb{S} = \begin{bmatrix} 1 & -\frac{1}{2}u'_n & \frac{u'_n(Q_n - 3Q_{n-1})}{2u_n} - \frac{Q_{n-1}}{2} & \frac{u'_n(Q_{n-1} - 3Q_{n-2})}{2u_n} - \frac{Q_{n-2}}{2} \\ 0 & 1 & -\frac{Q_n - 3Q_{n-1}}{u_n} & -\frac{Q_{n-1} - 3Q_{n-2}}{u_n} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dots & \frac{u'_n(Q_3 - 3Q_2)}{2u_n} - \frac{Q_2}{2} & & \\ \dots & -\frac{Q_3 - 3Q_2}{u_n} & & \\ \dots & 0 & & \\ \dots & 0 & & \\ \ddots & \vdots & & \\ \dots & 0 & & \\ \dots & 1 & & \end{bmatrix}$$

and $G = \text{diag}(2, u_n)$ where $u_n = Q_1 - 3Q_n + \sum_{k=2}^{n-1} (Q_{k+1} - 3Q_k) \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k}$

and $u'_n = \sum_{k=2}^n Q_k \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k}$. Then we obtain

$$K\mathbb{Q}L\mathbb{S} = G \oplus A$$

where $G \oplus A$ is the direct sum of the matrices G and A . If $W = L\mathbb{S}$, then we have

$$\mathbb{Q}^{-1} = W(G^{-1} \oplus A^{-1})K.$$

Since the matrix \mathbb{Q} is circulant, the inverse matrix \mathbb{Q}^{-1} is circulant from Lemma 1.1 [1, p. 9791]. Let

$$\mathbb{Q}^{-1} = \text{circ}(q_1, q_2, \dots, q_n).$$

Since the last row of the matrix W is

$$\left(0, 1, -\frac{Q_n - 3Q_{n-1}}{u_n}, -\frac{Q_{n-1} - 3Q_{n-2}}{u_n}, -\frac{Q_{n-2} - 3Q_{n-3}}{u_n}, \dots, -\frac{Q_4 - 3Q_3}{u_n}, -\frac{Q_3 - 3Q_2}{u_n}\right),$$

the last row elements of the matrix \mathbb{Q}^{-1} are

$$\begin{aligned} q_2 &= \frac{1}{u_n} \left(-3 + \sum_{k=1}^{n-2} (Q_{n-k+1} - 3Q_{n-k}) \left(\frac{(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \right) \\ q_3 &= -\frac{Q_3 - 3Q_2}{u_n(Q_1 - Q_{n+1})} \\ q_4 &= \frac{2(Q_3 - 3Q_2)}{u_n(Q_1 - Q_{n+1})} - \frac{1}{u_n} \sum_{k=1}^2 \frac{(Q_{5-k} - 3Q_{4-k})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \\ q_5 &= \frac{1}{u_n} \left(\frac{Q_3 - 3Q_2}{(Q_1 - Q_{n+1})} - \sum_{k=1}^3 \frac{(Q_{6-k} - 3Q_{5-k})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right. \\ &\quad \left. + 2 \sum_{k=1}^2 \frac{(Q_{5-k} - 3Q_{4-k})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \\ &\vdots \\ q_n &= -\frac{1}{u_n} \left(\sum_{k=1}^{n-2} \frac{(Q_{n-k+1} - 3Q_{n-k})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right. \\ &\quad \left. - 2 \sum_{k=1}^{n-3} \frac{(Q_{n-k} - 3Q_{n-k-1})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right. \\ &\quad \left. - \sum_{k=1}^{n-4} \frac{(Q_{n-k-1} - 3Q_{n-k-2})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \\ q_1 &= \frac{1}{u_n} \left(1 - \frac{8(Q_n - 2)^{n-3}}{(Q_1 - Q_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{(Q_{n-k+2} - 3Q_{n-k+1})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right) \end{aligned}$$

where $u_n = Q_1 - 3Q_n + \sum_{k=2}^{n-1} (Q_{k+1} - 3Q_k) \left(\frac{Q_n - 2}{Q_1 - Q_{n+1}} \right)^{n-k}$. Now, let us rearrange the q_i 's ($i \geq 4$). Hence we obtain

$$\begin{aligned} q_4 &= \frac{1}{u_n} \left(\frac{2Q_3 - 6Q_2 - Q_4 + 3Q_3}{Q_1 - Q_{n+1}} - \frac{(Q_3 - 3Q_2)(Q_n - 2)}{(Q_1 - Q_{n+1})^2} \right) \\ &= \frac{1}{u_n} \left(\frac{\overbrace{-Q_4 + 2Q_3 + Q_2}^0 + \overbrace{3Q_1 - Q_2}^0}{Q_1 - Q_{n+1}} - \frac{-4(Q_n - 2)}{(Q_1 - Q_{n+1})^2} \right) \\ &= \frac{4(Q_n - 2)}{u_n(Q_1 - Q_{n+1})^2} \end{aligned}$$

$$\begin{aligned} q_5 &= \frac{1}{u_n} \left[\frac{1}{Q_1 - Q_{n+1}} \left(\underbrace{(-Q_5 + 2Q_4 + Q_3)}_0 + 3 \underbrace{(Q_4 - 2Q_3 - Q_2)}_0 \right) \right. \\ &\quad \left. + \frac{Q_n - 2}{(Q_1 - Q_{n+1})^2} \left(\underbrace{(-Q_4 + 2Q_3 + Q_2)}_0 + \underbrace{3Q_1 - Q_2}_0 + \frac{4(Q_n - 2)^2}{(Q_1 - Q_{n+1})^3} \right) \right] \\ &= \frac{4(Q_n - 2)^2}{(Q_1 - Q_{n+1})^3}. \end{aligned}$$

If we formulate general case $i \geq 3$, then we have

$$q_i = \frac{4(Q_n - 2)^{n-i}}{(Q_1 - Q_{n+1})^{n-i}}.$$

Thus

$$\begin{aligned} \mathbb{Q}^{-1} &= \frac{1}{u_n} \text{circ} \left(1 - \frac{8(Q_n - 2)^{n-3}}{(Q_1 - Q_{n+1})^{n-2}} + \sum_{k=1}^{n-3} \frac{(Q_{n-k+2} - 3Q_{n-k+1})(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k}, \right. \\ &\quad \left. -3 + \sum_{k=1}^{n-2} (Q_{n-k+1} - 3Q_{n-k}) \left(\frac{(Q_n - 2)^{k-1}}{(Q_1 - Q_{n+1})^k} \right), \frac{4}{Q_1 - Q_{n+1}}, \right. \\ &\quad \left. \frac{4(Q_n - 2)}{(Q_1 - Q_{n+1})^2}, \frac{4(Q_n - 2)^2}{(Q_1 - Q_{n+1})^3}, \dots, \frac{4(Q_n - 2)^{n-3}}{(Q_1 - Q_{n+1})^{n-2}} \right) \end{aligned}$$

by the nice properties fo Pell-Lucas numbers. ■

Corollary 10 *Let the matrix K be as in (7). Then its inverse is block matrix*

$$K^{-1} = \begin{bmatrix} 1 & 0 \\ D & E \end{bmatrix}.$$

where the matrices

$$D = \left(\begin{array}{cccc} \frac{Q_n}{2} & \frac{Q_{n-1}}{2} & \frac{Q_{n-2}}{2} & \dots & \frac{Q_2}{2} \end{array} \right)_{(n-1) \times 1}^T$$

and E is an $(n-1) \times (n-1)$ Hankel matrix that it has the first row $[P_{n-1}, P_{n-2}, \dots, P_1]$ and the last column $[P_1, 0, \dots, 0]^T$.

Proof. The matrix K^{-1} is obtained easily by applying elementary row operations to the augmented matrix $[K : I_n]$. ■

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